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$$\begin{aligned}(l_2-1)a_1+m_2a_3+n_2a_4+a_2 &= 0,\\ (l_2-1)c_1+m_2c_3+n_2c_4+c_2 &= 0,\\ (l_2-1)d_1+m_2d_3+n_2d_4+d_2 &= 0.\end{aligned}$$

By this transformation the determinant is changed into

$$\begin{vmatrix} a_1 & a_1 & c_1 & d_1 \\ a_1 & a_1 & c_1 & d_1 \\ a_3 & a_3 & c_3 & d_3 \\ a_4 & a_4 & c_4 & d_4 \end{vmatrix}$$

a determinant in which the first two rows are identical and also the first two columns. The same process evidently applies to determinants of any order.

NEW NOTATION FOR ANHARMONIC RATIOS.

BY PROF. WILLIAM WOOLSEY JOHNSON.

1. THE notation here proposed has for its object to express in a symmetrical manner the relation between the six distinct values of the anharmonic ratio of four points and the four modes of writing each which constitute the 24 arrangements of the four letters concerned.

2. The anharmonic ratio of the section of AB by PQ is the ratio

$$\frac{AP}{BP} : \frac{AQ}{BQ};$$

regarding A and B as fixed points of reference, the constituent ratios $AP \div BP$ and $AQ \div BQ$ may be called the *position ratios* of P and Q. A distinction of sign being made between AP and PA, each value of the position ratio determines the position of a point, negative position ratios corresponding to points between A and B; while the position ratio of A is zero and that of B is infinity. The anharmonic ratio is the position ratio of P divided by that of Q, and regarding Q as a third fixed point of reference the value of the anharmonic ratio is a fixed multiple of the position ratio of P, and may be considered a coordinate determining the position of P, in such a manner that the coordinate of Q is unity.

3. Now let this anharmonic ratio be denoted by writing the four letters in a square form thus,

$$\begin{matrix} P & A \\ B & Q \end{matrix} = \frac{AP}{BP} : \frac{AQ}{BQ} = \frac{AP.BQ}{BP.AQ} = x, \qquad (1)$$

in which it is to be remembered that the letters occupy the position given to their coordinates in the form

$$\begin{matrix} x & 0 \\ \infty & 1 \end{matrix}$$

Equation (1) shows that with this notation the anharmonic ratio is *the product of the distances expressed by the rows divided by the product expressed by the columns,* care being taken to read the letters *similarly* in the two cases; that is, if in oppsite directions in the two rows, then also in opposite directions in the two columns, as in equation (1), but if in the same direction in the two rows then in the same direction in the two columns; thus,

$$\begin{matrix} P & A \\ B & Q \end{matrix} = \frac{AP.BQ}{BP.AQ} = \frac{PA.BQ}{PB.AQ}$$

4. It follows at once, from the rule, that an interchange of A and B or an interchange of P and Q converts the anharmonic ratio x into its reciprocal since it changes rows into columns and columns into rows. That is *the interchange of the letters forming either diagonal produces the reciprocal.* Simultaneous interchanges in both diagonals therefore do not affect the value, and in general, *simultaneous interchanges in two pairs of letters* (that is, in both rows, or in both columns, as well as in both diagonals) *do not alter the value of the ratio,* since the rows thus remain rows and the columns remain columns. Thus the four modes of writieg the same anharmonic ratio are

$$\begin{matrix} P & A \\ B & Q \end{matrix} = \begin{matrix} Q & B \\ A & P \end{matrix} = \begin{matrix} A & P \\ Q & B \end{matrix} = \begin{matrix} B & Q \\ P & A \end{matrix} \qquad (2)$$

5. The six distinct values of the anharmonic ratio of the four points may, as is well known, be obtained one from another by the process of alternately taking the reciprocal, and taking the *complement* (defining as complements quantities whose sum is unity). Thus, starting with x, the six values are

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ x & \dfrac{1}{x} & \dfrac{x-1}{x} & \dfrac{x}{x-1} & \dfrac{1}{1-x} & 1-x\,; \end{array}$$

the seventh term of the series being identical with the first. There are thus three pairs of reciprocals, or values having the symmetrical relation

$$xy = 1;$$

and also, otherwise grouped, there are three pairs of complements having the symmetrical relation

$$x + z = 1;$$

but furthermore, there are three pairs having the relation borne by the fourth term of the series to the first; viz.,

$$t = \frac{x}{x-1},$$

or

$$xt = x + t,$$

which is also a symmetrical relation. For want of a more distinctive term, I shall call two quantities bearing this relation *conjugates.*

6. If we denote the operation of taking the reciprocal by R, that of taking the complement by C, and that of taking the conjugate by J, we have

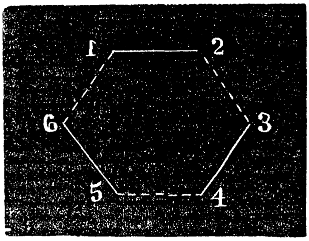
$$R.Rx = x \qquad C.Cx = x \qquad J.Jx = x,$$

or symbolically,

$$R^2 = 1 \qquad C^2 = 1 \qquad J^2 = 1. \tag{3}$$

Placing the six anharmonic ratios at the vertices of a hexagon as in the diagram, R is the operation of passing in either direction along one of the full-lined sides, C that of passing along one of the dotted sides, and J that of passing either way along one of the diagonals. It is obvious that we have

$$\left.\begin{aligned} CRC &= RCR = J, \\ JCJ &= CJC = R, \\ RJR &= JRJ = C. \end{aligned}\right\} \tag{4}$$



7. We shall now show that in the proposed notation each of the operations C and J, as well as the operation R, may be effected by a simple interchange. Consider first the effect of interchanging the letters in one of the columns of the square form x, and let

$$z = \begin{matrix} B & A \\ P & Q \end{matrix} = \frac{AB\,.\,PQ}{PB\,.\,AQ}$$

in which the denominator is the negative of that of x in equation (1).

We have then

$$x + z = \frac{AP\,.\,BQ - AB\,.\,PQ}{BP\,.\,AQ};$$

denoting for the moment AB by a, BP by b, and PQ by c,

$$AP = a + b, \qquad BQ = b + c,$$

and

$$AQ = a + b + c,$$

whence substituting we have

$$x + z = \frac{(a+b)(b+c) - ac}{b(a+b+c)} = 1$$

and z is the complement of x. An interchange in the other column has a like effect, since, as shown in 4, the two interchanges reproduce x. Thus *an interchange of the letters in either column produces the complement.*

8. If in the form $x = \begin{matrix} P & A \\ B & Q \end{matrix}$

we allow one letter to remain stationary, the other three letters form a triangle, and it is readily seen that an interchange along one side of the triangle followed by an interchange along a second side and another interchange along the first side restores the letter which was at the common vertex of these sides and is therefore equivalent to a single interchange along the third side. Thus allowing Q to remain stationary in the value of x, and making the interchanges represented by R, C and R in this order, we have

$$\begin{aligned} Rx &= \begin{matrix} P & B \\ A & Q \end{matrix} \\ CRx &= \begin{matrix} A & B \\ P & Q \end{matrix} \\ RCRx &= \begin{matrix} A & P \\ B & Q \end{matrix} \end{aligned}$$

which might have been obtained by interchanging the letters of the upper row. Now it is shown in 6 that $RCRx = Jx$; hence this interchange has the effect of taking the conjugate, and since interchanges in both rows reproduce x, we see that *an interchange of the letters in either row produces the conjugate.*

9. The operations R, C and J respectively convert x into three of the other five values of the anharmonic ratio; the other two values are the complement of the reciprocal

$$CRx = \frac{x-1}{x},$$

and the reciprocal of the complement

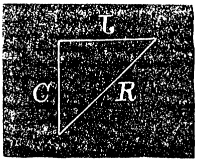
$$RCx = \frac{1}{1-x}.$$

We readily derive from the hexagonal arrangement in 6, or from equations (3) and (4), the identities

$$CR = RJ = JC, \tag{5}$$
$$RC = CJ = JR. \tag{6}$$

Each of these operations is such that thrice repeated it reproduces x, and each is the inverse of the other; (5) is equivalent to passing in the direction of the hands of a watch along one of the sides of the triangle 2 4 6, or in the opposite direction along one of the sides of 1 3 5.

10. Recurring to the square form $x = \begin{matrix} P & A \\ B & Q \end{matrix}$, it will be found that *the effect of two unlike interchanges* as in (5) or (6) *is to produce a cyclic displacement of three of the letters* (the fourth remaining in its place); and if we mark the sides of the triangle formed by these three letters R, C and J, *the direction of the displacement is the same as that in which the letters of the compound symbol occur on this triangle.* For example, if Q is the stationary letter, the triangle is as in the diagram;



the symbols in (5) occur on this triangle in the direction contrary to that of the hands of a watch, and the displacement of the letters takes place in this direction; thus

$$CRx = \begin{matrix} A & B \\ P & Q \end{matrix}.$$

On the other hand

$$RCx = \begin{matrix} B & P \\ A & Q \end{matrix}.$$

11. In comparing any two of the 24 square arrangements of the four letters, one of the letters may by the process indicated in equation (2) be brought into the same position in each form; the forms will then differ either in the position of two of the remaining letters indicating one of the relations denoted by R, C or J or else in the position of all three letters indicating one of the relations denoted by CR or RC, discussed above.

Solution of Prob. 407 by prof. Edgar Frisby.—I notice that the solution of problem 407 [see p. 158] contains a remarkable mistake in omitting the exponent $\frac{1}{2}$. The expression is an elliptic integral; it can be reduced to a series thus:

$$\int_0^1 \frac{(1+x^4)^{\frac{1}{2}}}{(1-x)^{\frac{1}{2}}}\,dx = \int_0^1 \frac{1+\frac{1}{2}x^4-\frac{1}{2}.\frac{1}{4}x^8+\frac{1}{2}.\frac{1}{4}\frac{3}{6}x^{12}-\frac{1}{2}.\frac{1}{4}.\frac{3}{6}\frac{5}{8}x^{16}+-}{\sqrt{(1-x^2)}}\,dx.$$

Integrating each term by parts and only retaining the last term of each series, as the other terms vanish at both limits, we have

$$\int_0^{\frac{\pi}{2}} (1+\cos^4\theta)^{\frac{1}{2}}d\theta = \frac{\pi}{2}\left\{ \frac{1}{2}\frac{1.3}{2.4}-\frac{1.1}{2.4}\frac{1.3.57}{2.4.68}+\frac{1.1.3}{2.4.6}\frac{1.3.5.7.9.11}{2.4.6.8.10.12}\cdots\right\}.$$

SOLUTIONS OF PROBLEMS IN NUMBER FIVE.

Solutions of problems in No. 5 have been received as follows:

From Prof. L. G. Barbour, 412; Alex. S. Christie, 411; Prof, W. P. Casey, 409, 412, 416; Geo. Eastwood, 413, 416, 417; C. E. Everett, 409, 410, 415, 416; Prof. Edgar Frisby, 409, 411, 415; Wm. Hoover, 409, 411, 413 416; L. S. Hulbert, 412; E. H. Moore, Jr., 416; Prof. P. H. Philbrick, 409, 414, 415, 416; A. L. Parman, 409; Prof. E. B. Seitz, 409; Prof. J. Scheffer, 409, 410, 411, 416; Prof. C. M. Woodward, 409 410, 413; R. S. Woodward, 413.

[Solutions of 401, 402, 403 and 406 by Chas. E. Everett and Prof. Casey, respectively, were received too late for acknowledgment in No. 5.]